# The characterisation of the smallest two fold blocking sets in PG$(n, 2)$

Manohar L. Aggarwal, Andreas Klein and Leo Storme

July 4, 2011

**Abstract**

We classify the smallest two fold blocking sets with respect to the $(n-k)$-spaces in PG$(n, 2)$. We show that they either consist of two disjoint $k$-dimensional subspaces or are equal to a $(k+1)$-dimensional space minus one point.

## 1 Introduction

A *blocking set with respect to the codimension $k$ spaces* of PG$(n, q)$ is a set $B$ of points intersecting every codimension $k$ space in at least one point.

A blocking set with respect to the codimension $k$ spaces is called *minimal* when no proper subset of $B$ still is a blocking set with respect to the codimension $k$ spaces.

A famous theorem of Bose and Burton states that the smallest blocking sets in PG$(n, q)$ are subspaces.

**Theorem 1 (Bose and Burton [2])**
*Let $B$ be a blocking set of $PG(n, q)$ with respect to codimension $k$ spaces. Then $|B| \geq \frac{q^{k+1}-1}{q-1}$ with equality if and only if $B$ is a $k$-dimensional subspace of $PG(n, q)$.*

A blocking set $B$ with respect to the codimension $k$ spaces is called *non-trivial* when it does not contain a $k$-dimensional subspace.

The smallest non-trivial blocking sets are characterised by Beutelspacher and Heim. Let $q + r(q) + 1$ be the size of the smallest non-trivial blocking sets in PG$(2, q)$.

**Theorem 2 (Beutelspacher and Heim [1, 4])**
*For a non-trivial blocking set $B$ in $PG(n, q)$, $q > 2$, with respect to $k$-subspaces, $|B| \geq q^{n-k} + r(q)q^{n-k-1} + q^{n-k-1} + q^{n-k-2} + \cdots + q + 1$ and $|B| = q^{n-k} + r(q)q^{n-k-1} + q^{n-k-1} + q^{n-k-2} + \cdots + q + 1$ if and only if $B$ is equal to a cone with an $(n - k - 2)$-dimensional vertex and base a minimal non-trivial blocking set of size $q + r(q) + 1$ in a plane skew to the vertex.*

The case $q = 2$ differs from the case $q > 2$ in the sense that the plane $\mathrm{PG}(2, 2)$ does not contain a non-trivial blocking set. This explains the condition $q > 2$ in the preceding theorem.

Govaerts and Storme characterised the smallest non-trivial blocking sets with respect to the planes and the lines in $\mathrm{PG}(3, 2)$.

**Theorem 3 (Govaerts and Storme [3])**

1. *In $PG(n, 2)$, $n \geq 3$, the smallest non-trivial blocking sets with respect to hyperplanes are skeletons of solids in $PG(n, 2)$; these are sets of five points in a 3-space, no four of which are coplanar. If $n = 3$, then these are the only minimal non-trivial blocking sets with respect to planes. So, up to isomorphism, there is only one non-trivial minimal blocking set with respect to planes in $PG(3, 2)$.*

2. *Up to isomorphism, there is only one non-trivial minimal blocking set with respect to lines in $PG(3, 2)$. It consists of ten points and is the set of points on the edges of a tetrahedron.*

They then generalised this theorem to the following general theorem.

**Theorem 4 (Govaerts and Storme [3])**
*In $PG(n, 2)$, $n \geq 3$, the smallest non-trivial blocking sets with respect to $t$-spaces, $1 \leq t \leq n - 2$, have size $2^{n-t+1} + 2^{n-t-1} + 2^{n-t-2} - 1$ and are cones with vertex an $(n - t - 3)$-space $\pi_{n-t-3}$ and base the set of points on the edges of a tetrahedron in a solid skew to $\pi_{n-t-3}$.*

One way to generalise this result is to ask for characterisations of multiple blocking sets. For instance, a *two fold (or 2-fold) blocking set with respect to the codimension $k$ spaces* is a set $B$ of points in $\mathrm{PG}(n, q)$ intersecting every codimension $k$ space in at least two points.

One obvious way to construct a two fold blocking set is to take two disjoint $k$-dimensional subspaces in $\mathrm{PG}(n, q)$ if $2k + 1 \leq n$.

For $q = 2$, there exists an other example. One can take a $(k+1)$-dimensional space and remove one point. Each codimension $k$ space intersects the $(k + 1)$-dimensional space in at least a line. If this line goes through the "missing point", it still contains two points of $B$, otherwise it will contain even three points of $B$.

The goal of this article is to classify the smallest 2-fold blocking sets of $\mathrm{PG}(n, 2)$ and to show that the two examples above are indeed the only smallest ones.

We first prove some auxiliary results.

## 2 3-Fold blocking sets

**Theorem 5**
*A 3-fold blocking set of $PG(n,2)$ with respect to codimension $k$ spaces must contain at least $2^{k+2}-1$ points with equality if and only if it consists of the points of a $(k+1)$-dimensional subspace.*

**Proof**
Let $B$ be a 3-fold blocking set of size at most $2^{k+2}-1$. If $B$ blocks every codimension $k+1$ space, then by the Bose-Burton theorem, $B$ must be a $(k+1)$-dimensional space.

If $B$ does not block every codimension $k+1$ space, then let $\pi$ be a codimension $k+1$ space that contains no points of $B$.

Each of the $2^{k+1}-1$ codimension $k$ spaces through $\pi$ must contain at least 3 points of $B$ and hence $|B| \geq 3(2^{k+1}-1)$.

## 3 4-Fold blocking sets with respect to hyperplanes

**Theorem 6**
*A 4-fold blocking set $B$ of $PG(n,2)$ with respect to hyperplanes must contain at least 10 points. The only 4-fold blocking sets of size 10 are:*

- *A plane together with a skew line.*
- *A 3-space without the 5 points of a 5-arc. Equivalently, this is the set of 10 points on the edges of a tetrahedron.*

**Proof**
Let $|B| \leq 10$. The set $B$ must block all codimension 2 spaces or otherwise $|B| \geq 4 \cdot 3 = 12$ since all the three hyperplanes through a codimension 2 space skew to $B$ need at least 4 points of $B$.

Hence, by Theorem 4, $B$ must either contain a plane or the 10 points of $B$ must lie on the edges of a tetrahedron.

In the second case, the points of $B$ form a 3-space without the points of a 5-arc.

In the first case, $B$ is of the form of a plane plus 3 extra points. Suppose that these 3 extra points are not collinear.

Assume first of all that they define a plane $\pi'$ skew to $\pi$. Take a line $\ell'$ in $\pi'$ skew to the three extra points and take a line $\ell$ in $\pi$. Then there is at least one hyperplane through the 3-space $\langle \ell, \ell' \rangle$ only sharing the lines $\ell$ and $\ell'$ with $\pi$ and $\pi'$. But then this hyperplane only has three points in common with $B$. This is a contradiction.

Assume now that the two planes $\pi$ and $\pi'$ share one point. Let $\ell'$ be a line skew to $B$ in $\pi'$. If $\ell'$ is skew to $\pi$, then take a line $\ell$ in $\pi$. The same arguments as in the preceding case lead to the same contradiction. If $\ell'$ intersects $\pi$, then take a line $\ell$ in $\pi$ through the point $\pi \cap \pi'$. Then the plane $\langle \ell, \ell' \rangle$ lies in at least one hyperplane only sharing the lines $\ell$ and $\ell'$ with $\pi$ and $\pi'$. Then this hyperplane again only contains three points of $B$. This is false.

Finally, if the planes $\pi$ and $\pi'$ share a line, it is possible to find a hyperplane through this intersection line, not containing $\pi$ nor $\pi'$. Then again this hyperplane contains three points of $B$. This is false. $\square$

**Remark 1**
*The preceding result was also proven in the context of minihypers and their relation to linear codes meeting the Griesmer bound. An* $\{f, m; n, q\}$-minihyper *$F$ is a set of $f$ points in PG$(n, q)$ intersecting every hyperplane in at least $m$ points, where at least one hyperplane contains exactly $m$ points of $F$.*

*A $\{\sum_{i=1}^{p}(2^{u_i} - 1), \sum_{i=1}^{p}(2^{u_i-1} - 1); k - 1, 2\}$-minihyper, with $k > u_1 > \cdots > u_p \geq 1$, is equivalent to a linear $[n = 2^k - 1 - \sum_{i=1}^{p}(2^{u_i} - 1), k, d = 2^{k-1} - \sum_{i=1}^{p} 2^{u_i-1}]$-linear binary code, meeting the Griesmer bound.*

*Helleseth gave a complete characterisation of these linear codes, so equivalently, of the corresponding minihypers [5]. The 4-fold blocking set of size 10 in PG$(n, 2)$, $n \geq 3$, discussed in Theorem 6, is a particular example of such a minihyper, namely a $\{10, 4; n, 2\}$-minihyper.*

# 4 2-Fold blocking sets

## 4.1 A lower bound on the size of a 2-fold blocking set in PG$(n, 2)$

We start our analysis with the proof of a lower bound on the size of a 2-fold blocking set in PG$(n, 2)$.

Let $\theta_k = 2^{k+1} - 1$ be the number of points in PG$(k, 2)$.

**Theorem 7**
*A set $B$ in PG$(n, 2)$ blocking all codimension $k$ spaces twice contains at least $2\theta_k$ points.*

**Proof**
Let $B$ be a two fold blocking set with at most $2\theta_k$ points. Since $\theta_{k+1} > 2\theta_k$, the Bose-Burton theorem (Theorem 1) shows that $B$ cannot block all codimension $k + 1$ spaces. Let $\pi$ be a codimension $k + 1$ space which contains no points of $B$, then all $\theta_k$ codimension $k$ spaces through $\pi$ must contain two points outside of $\pi$, which proves that $B$ has at least $2\theta_k$ points. $\square$

From now on, we assume that $B$ is a two fold blocking set with respect to codimension $k$ spaces of size $2\theta_k$.

We will prove the classification theorem by induction on $k$ starting with the two cases $k = 1$ and $k = 2$.

## 4.2 Blocking sets with respect to hyperplanes

**Lemma 1**
*A double blocking set $B$ in $PG(3,2)$ contains at least 6 points. If $B$ is a double blocking set of 6 points in $PG(3,2)$, then $B$ contains a line.*

**Proof**
If $B$ also blocks every line, then Theorem 1 states that the size of $B$ is at least seven. So, from now on, assume that $\ell$ is a line skew to $B$. Then the three planes through $\ell$ all contain at least two points of $B$, so the size of $B$ is at least six.

Suppose that $B$ has size 6 and does not contain a line, then $B$ is a 6-cap. The complete caps in $\mathrm{PG}(3,2)$ are classified [6]. There are only two examples:

- a skeleton (5 points projectively equivalent to $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,1,1,1)$),
- the complement of a plane.

Since $B$ is a cap of 6 points, it must be contained in the complement of a plane; a contradiction to the assumption that $B$ is a double blocking set. $\square$

From now on, in Lemmas 2 to 4, let $B$ be a double blocking set of size 6 in $\mathrm{PG}(3,2)$.

**Lemma 2**
*Let $\ell$ be a line contained in $B$. Let $R \in \ell$ and let $\pi$ be a plane not through $R$. The projection of $B \setminus \{R\}$ from $R$ to $\pi$ contains a line $h$.*

**Proof**
A plane through $R$ contains at least two points of $B$, hence the projection of $B \setminus \{R\}$ from $R$ to $\pi$ must be a line blocking set of $\pi$. Every blocking set of $\mathrm{PG}(2,2)$ contains a line $h$. $\square$

We now use the notations of the preceding lemma.

**Lemma 3**
*If the line $h$ does not pass through the point $\ell \cap \pi$, then $B$ consists of two skew lines.*

**Proof**
Let $R_1$, $R_2$ and $R_3$ be the three points of $B \setminus \ell$. The three points $R_1, R_2, R_3$ lie in the plane $\langle R, h\rangle$. For every line $m$ in this plane $\langle h, R\rangle$, we find a plane through it only sharing one point with $\ell$. Then this implies that every line of $\langle h, R\rangle$ not through $R$ contains at least one of the points $R_1, R_2, R_3$ and that every line of $\langle h, R\rangle$ through $R$ also contains at least one of the points $R_1, R_2, R_3$. So, the three points $R_1, R_2, R_3$ form a blocking set in $\langle h, R\rangle$. By the Bose-Burton theorem (Theorem 1), they form a line in $\langle h, R\rangle$. $\square$

**Lemma 4**
*If the line $h$ passes through $\ell \cap \pi$, then all points of $B$ lie in a plane.*

**Proof**
The plane $\langle \ell, h \rangle$ contains at least 5 points of $B$. Assume that it does not contain all six points of $B$. Then a certain line of $\langle \ell, h \rangle$ contains only one point of $B$ and since all planes through that line must contain two points of $B$, the set $B$ would have at least $5 + 2$ points. This is false. $\square$

At this point, we have classified the two fold blocking sets of PG$(3, 2)$ of size 6; we now extend the result to arbitrary dimensions.

**Theorem 8**
*Let $B$ be a two fold blocking set of 6 points in PG$(n, 2)$ with respect to the hyperplanes. Then $B$ consists either of two skew lines or all points of $B$ lie in a plane.*

**Proof**
If the 6 points span a space PG$(5, 2)$, then they are projectively equivalent to $(1, 0, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0, 0)$,..., $(0, 0, 0, 0, 0, 1)$, but this is not a double blocking set. If the 6 points span a space PG$(4, 2)$, then, up to projective equivalence, $B$ contains the points $(1, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0)$,..., $(0, 0, 0, 0, 1)$. The hyperplane $x_0 + x_1 + x_2 + x_3 + x_4 = 0$ contains none of these points. Thus to become a two fold blocking set, $B$ needs at least 2 more points, but only one is left.

Thus $B$ is contained in a 3-space and must form a two fold blocking set with respect to the hyperplanes of that 3-space. By the preceding lemmas, we conclude that either the 6 points of $B$ must form two skew lines or lie in a plane. $\square$

### 4.3 Blocking sets with respect to codimension 2 spaces

**Lemma 5**
*Let $B$ be a two fold blocking set with respect to the codimension two spaces, then every hyperplane contains at least 6 points of $B$.*

**Proof**
Assume otherwise, then, by Theorem 1, the hyperplane contains a codimension 3 space $\pi$ with no point in $B$. The three codimension 2 spaces through $\pi$ inside the hyperplane contain together at least $3 \cdot 2 = 6$ points of $B$. $\square$

From now on, in Lemmas 6 to 10, let $B$ be a two fold blocking set of size 14 with respect to the codimension 2 spaces.

The goal of the next lemmas is to prove that a hyperplane with more than 6 points of $B$ must either share at least a plane with $B$, or must contain all the points of $B$.

**Lemma 6**
*A hyperplane $\Pi$ of PG$(n, 2)$ cannot contain exactly 9 points of $B$.*

**Proof**
Each codimension 2 space of PG$(n, 2)$ inside $\Pi$ must meet $B$ in 4 points or

otherwise, by counting the number of points of $B$ in the three hyperplanes through this codimension 2 space, $B$ would contain at least $3+(9-3)+2\cdot(6-3)=15$ points. Suppose that there is a codimension 3 space of $\mathrm{PG}(n,2)$ inside $\Pi$ which contains only one point of $B$. Counting the codimension 2 spaces inside $\Pi$ through that codimension 3 space shows that $\Pi$ contains at least $1+3\cdot(4-1)=10$ points of $B$. Thus every codimension 3 space of $\mathrm{PG}(n,2)$ inside $\Pi$ is blocked at least twice which shows, by Theorem 7, that $\Pi$ must contain all 14 points of $B$. $\square$

**Lemma 7**
*A hyperplane $\Pi$ of $PG(n,2)$ cannot contain exactly 11 or exactly 12 points of $B$.*

**Proof**
If a hyperplane $\Pi$ contains more than 10 points of $B$, then every codimension 2 space of $\mathrm{PG}(n,2)$ inside $\Pi$ must contain at least 5 points of $B$ or otherwise $B$ would consist of at least $4+(11-4)+2\cdot(6-4)=15$ points. Then every codimension 3 space of $\mathrm{PG}(n,2)$ inside $\Pi$ must contain at least 2 points of $B$ or otherwise $\Pi$ would contain at least $1+3\cdot(5-1)=13$ points of $B$. So $B\cap\Pi$ is a two fold blocking set with respect to the codimension 2 spaces in $\Pi$, i.e. $\Pi$ contains at least 14 points of $B$. $\square$

**Lemma 8**
*A hyperplane $\Pi$ of $PG(n,2)$ cannot contain exactly 13 points of $B$.*

**Proof**
If a hyperplane $\Pi$ contains 13 points of $B$, then every codimension 2 space of $\mathrm{PG}(n,2)$ inside $\Pi$ must contain at least 6 points of $B$, or otherwise $B$ would consist of at least $5+(13-5)+2\cdot(6-5)=15$ points. Thus every codimension 3 space of $\mathrm{PG}(n,2)$ inside $\Pi$ must contain at least 2 points of $B$ or otherwise $\Pi$ would contain at least $1+3\cdot(6-1)=16$ points of $B$. So $B\cap\Pi$ is a two fold blocking set, i.e. $\Pi$ contains at least 14 points of $B$. $\square$

**Lemma 9**
*A hyperplane $\Pi$ of $PG(n,2)$ which meets $B$ in exactly 7 or exactly 8 points contains a plane entirely inside $B$.*

**Proof**
If a hyperplane $\Pi$ contains 7 or 8 points of $B$, then every codimension 2 space of $\mathrm{PG}(n,2)$ inside $\Pi$ must contain at least 3 points of $B$ or otherwise $B$ would consist of at least $2+(7-2)+2\cdot(6-2)=15$ points.

This implies that every codimension 3 space of $\mathrm{PG}(n,2)$ inside $\Pi$ is blocked by $B$ or otherwise $\Pi$ would contain at least $3\cdot 3=9$ points of $B$.

Since $B\cap\Pi$ is a blocking set with respect to codimension 3 spaces inside $\Pi$ of size 7 or 8, it follows that $B\cap\Pi$ must contain a plane (Theorems 3 and 4). $\square$

**Lemma 10**
*If a hyperplane $\Pi$ of $PG(n,2)$ contains exactly 10 points of $B$, then $B \cap \Pi$ is equal to the union of a plane and a line which are skew to each other.*

**Proof**
Each codimension 2 space of $\mathrm{PG}(n,2)$ inside $\Pi$ must meet $B$ in at least 4 points, or else $B$ would consist of at least $3+(10-3)+2\cdot(6-3)=16$ points.

By Theorem 6, we know that $B \cap \Pi$ must be either a plane plus a skew line or $B \cap \Pi$ must be a 3-space minus the points of a 5-arc. This latter 5-arc is the skeleton of a tetrahedron.

In the last case, take a codimension 2 space of $\mathrm{PG}(n,2)$ inside $\Pi$ that meets $B$ in exactly 4 points. These four points form a line, plus an extra point $P$. This extra point $P$ is the third point on the edge of the tetrahedron, different from the vertices of the tetrahedron on this edge.

Let $\Pi'$ be another hyperplane through that codimension 2 space. Then $\Pi'$ contains exactly 6 points of $B$. These 6 points form a two fold blocking set in $\Pi'$ and hence, by Theorem 8, either $\Pi' \cap B$ are two skew lines or $\Pi' \cap B$ lies in a plane. The last case is impossible due to the construction of $\Pi'$.

So we have shown that the 4 points of $B$ not in $\Pi$ lie on two lines through $P$. But we can start the same argument with other codimension 2 spaces of $\mathrm{PG}(n,2)$ inside $\Pi$ which contain exactly 4 points of $B$, i.e. we find many points $P$ of $\Pi \cap B$ such that the 4 points of $B$ not in $\Pi$ lie on two lines through $P$. This implies that they then lie in a plane sharing at most a line with the 3-space containing the tetrahedron. But this is absurd since there are 6 different choices for the point $P$.

Hence, the only possible way to have a 4-fold blocking set of size 10 in $\Pi$ is to take a plane and a skew line. $\square$

**Theorem 9**
*Let $B$ be a two fold blocking set of size 14 with respect to codimension 2 spaces in $PG(n,2)$, then*

1. *$B$ consists of either two skew planes, or*

2. *all 14 points of $B$ lie in a 3-space.*

**Proof**
The proof goes by induction on $n$. For $n=3$, there is nothing to prove.

Now let $n>3$. By counting the average number of points in a hyperplane, we find a hyperplane $\Pi$ which contains more than 6 points of $B$. By the preceding lemmas, either $\Pi$ contains all points of $B$, and in that case we can apply induction, or $\Pi$ must contain a plane $\pi$ of points in $B$. In the latter case, we can find a hyperplane $\Pi'$, intersecting $\pi$ in a line, which contains more than 6 points of $B$ and again by the preceding lemmas we conclude that $\Pi'$ must contain a plane $\pi'$ of points in $B$.

If $\pi$ and $\pi'$ do not intersect, we have the case described by the theorem. If the two planes intersect in a line, then they define a 3-space containing already

11 points of $B$. This implies that there is a hyperplane containing at least 11 points of $B$; hence, by the preceding lemma, all points of $B$ lie in this hyperplane, and we can apply induction. If the two planes share one point, then take a 3-space containing one of those planes and one line of the other plane through the intersection point of the two planes. This 3-space then lies in a hyperplane containing at least 9 points of $B$. Either it contains all points of $B$ or exactly 10 points of $B$. But in this latter case, the intersection consists of a plane and a line skew to it, which is not the case here. The preceding lemmas show that $B$ must be contained inside that hyperplane and we can apply induction. $\square$

## 4.4 Blocking sets with respect to codimension $k$ spaces

From now on, let $B$ be a two fold blocking set with respect to codimension $k$ spaces of size $2\theta_k$.

**Lemma 11**
*Each hyperplane contains at least $2\theta_{k-1}$ points of $B$ and each codimension 2 space contains at least $2\theta_{k-2}$ points of $B$.*

**Proof**
Apply Theorem 7 to hyperplanes and codimension 2 spaces, respectively. $\square$

**Lemma 12**
*Let $\pi$ be a codimension 2 space which contains exactly $2\theta_{k-2}$ points, then every hyperplane through $\pi$ must contain exactly $2\theta_{k-1}$ points.*

**Proof**
Counting the points of $B$ in the hyperplanes through $\pi$, we find that $|B| \geq 2\theta_{k-2} + 3(2\theta_{k-1} - 2\theta_{k-2})$ with equality if and only if every hyperplane through $\pi$ contains exactly $2\theta_{k-1}$ points of $B$.

Since $|B| = 2\theta_k = 2\theta_{k-2} + 3(2\theta_{k-1} - 2\theta_{k-2})$, we have equality in the above inequality. $\square$

We now come to the main characterisation result of this article.

**Theorem 10**
*A two fold blocking set $B$ with respect to the codimension $k$ spaces of $PG(n, 2)$ contains at least $2\theta_k$ points. If $B$ contains exactly $2\theta_k$ points, then $B$ consists either of two skew $k$-spaces or all points of $B$ lie in a $(k+1)$-space.*

**Proof**
We prove the theorem by induction on $k$. In the previous sections, we have dealt with the cases $k = 1$ and $k = 2$. Assume now $k \geq 3$.

Let $\Delta$ be a codimension 2 space which intersects $B$ in $2\theta_{k-2}$ points. The existence of such a space $\Delta$ is proven in the following way. Since $|B| = 2\theta_k < \theta_{k+1}$, the Bose-Burton theorem (Theorem 1) implies that there exists a codimension $k+1$ space $\Pi_{n-k-1}$ skew to $B$. Every codimension $k$ space through $\Pi_{n-k-1}$ contains at least two points of $B$. Since there are exactly $\theta_k$ such codimension $k$

spaces and since $|B| = 2\theta_k$, these spaces necessarily contain precisely two points of $B$. This then implies that an arbitrary codimension two space $\Delta$ through $\Pi_{n-k-1}$ contains exactly $2\theta_{k-2}$ points of $B$.

By induction on $k$, the points of $B$ in $\Delta$ lie either in two skew $(k-2)$-spaces or in a $(k-1)$-space.

**Case 1:** assume that $\Delta \cap B$ is equal to a $(k-1)$-space minus one point.

Let $H_1, H_2, H_3$ be the three hyperplanes through $\Delta$. Then each hyperplane $H_i$ contains $2\theta_{k-1}$ points of $B$ (Lemma 12), and applying induction on $k$, we find that these $2\theta_{k-1}$ points of $B$ in $H_i$, $i = 1, 2, 3$, must form a $k$-space $\mathrm{PG}(k, 2)_i$ minus one point. Thus $B$ consists of three $k$-spaces through a common $(k-1)$-space, minus one point in this common $(k-1)$-space.

Select in $H_1$ a new $(n-2)$-dimensional space $\Delta'$ only sharing a $(k-1)$-space minus one point, different from $\Delta \cap B$, with $B$. Let $H_1' = H_1, H_2', H_3'$ be the three hyperplanes through $\Delta'$. They again share a $k$-space $\mathrm{PG}(k, 2)_i'$ minus one point with $B$. Here, $\mathrm{PG}(k, 2)_1 = \mathrm{PG}(k, 2)_1'$. Then $H_2'$ intersects $B \cap H_2$ in a $(k-1)$-dimensional space $\mathrm{PG}(k-1, 2)_2$ minus one point and intersects $B \cap H_3$ in a $(k-1)$-dimensional space $\mathrm{PG}(k-1, 2)_3$ minus one point, all lying in $\mathrm{PG}(k, 2)_2'$. This implies that the $(k+1)$-dimensional space $\langle \mathrm{PG}(k, 2)_1, \mathrm{PG}(k, 2)_2' \rangle$ contains all spaces $\mathrm{PG}(k, 2)_i$, $i = 1, 2, 3$.

This shows that $B$ is equal to a $(k+1)$-space minus one point.

**Case 2:** assume that $\Delta \cap B$ is equal to two disjoint $(k-2)$-spaces.

Let $H_1, H_2, H_3$ be the three hyperplanes through $\Delta$. Let $H_i \cap B$, $i = 1, 2, 3$, be the union of the two disjoint $(k-1)$-spaces $\mathrm{PG}(k-1, 2)_1^{(i)}$ and $\mathrm{PG}(k-1, 2)_2^{(i)}$, where the three spaces $\mathrm{PG}(k-1, 2)_1^{(i)}$ pass through the same $(k-2)$-space in $\Delta \cap B$ and also the three spaces $\mathrm{PG}(k-1, 2)_2^{(i)}$ pass through the same $(k-2)$-space in $\Delta \cap B$.

Select in $H_1$ a new $(n-2)$-dimensional space $\Delta'$ only intersecting $\mathrm{PG}(k-1, 2)_1^{(1)}$ and $\mathrm{PG}(k-1, 2)_2^{(1)}$ in two disjoint $(k-2)$-spaces, different from $\Delta \cap B$. Then the three hyperplanes $H_1 = H_1', H_2', H_3'$ intersect $B$ in two disjoint $(k-1)$-spaces $\mathrm{PG}(k-1, 2)_1^{(i)'}$ and $\mathrm{PG}(k-1, 2)_2^{(i)'}$, $i = 1, 2, 3$.

Then $H_2'$ intersects $\mathrm{PG}(k-1, 2)_1^{(2)}$ in a $(k-2)$-space and intersects $\mathrm{PG}(k-1, 2)_1^{(3)}$ in a $(k-2)$-space.

Assume that $\mathrm{PG}(k-1, 2)_1^{(1)}$ and $\mathrm{PG}(k-1, 2)_1^{(2)'}$ share a $(k-2)$-space. Then the preceding arguments show that the $k$-space $\langle \mathrm{PG}(k-1, 2)_1^{(1)}, \mathrm{PG}(k-1, 2)_1^{(2)'} \rangle$ contains the three $(k-1)$-spaces $\mathrm{PG}(k-1, 2)_1^{(i)}$, all passing through a common $(k-2)$-space. Hence, this shows that this $k$-space is contained in $B$.

A similar argument for the space $\langle \mathrm{PG}(k-1, 2)_2^{(1)}, \mathrm{PG}(k-1, 2)_2^{(2)'} \rangle$ shows that also this $k$-space is contained in $B$. We conclude that $B$ is the union of two disjoint $k$-spaces. $\square$

## References


[1] A. Beutelspacher, Blocking sets and partial spreads in finite projective spaces. *Geom. Dedicata* **9** (1980), 130-157.

[2] R.C. Bose and R.C. Burton, A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the McDonald codes. *J. Combin. Theory* **1** (1966), 96-104.

[3] P. Govaerts and L. Storme, The classification of the smallest nontrivial blocking sets of $PG(n, 2)$. *J. Combin. Theory, Ser. A* **113** (2006), 1543-1548.

[4] U. Heim, Blockierende Mengen in endlichen projektiven Räumen. (German) [Blocking sets in finite projective spaces] Dissertation, Justus-Liebig-Universität Giessen, Giessen, 1995. *Mitt. Math. Sem. Giessen* **226** (1996).

[5] T. Helleseth, A characterization of codes meeting the Griesmer bound. *Info. Control* **50** (1981), 128-159.

[6] J.W.P. Hirschfeld, *Finite projective spaces of three dimensions.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1985.



Address of the authors:

Manohar L. Aggarwal, The University of Memphis, Department of Mathematical Sciences, Memphis, TN 38152 (maggarwl@memphis.edu)

Andreas Klein, Ghent University, Dept. of Mathematics, Krijgslaan 281-S22, 9000 Ghent, Belgium (klein@cage.ugent.be)

Leo Storme, Ghent University, Dept. of Mathematics, Krijgslaan 281-S22, 9000 Ghent, Belgium (ls@cage.ugent.be, http://cage.ugent.be/~ls)